Tentamen
SOLID MECHANICS (NASM)
April 18, 2011, 14:00–17:00 h

Question 1

a. According to Mohr's circle, the three principal stresses $\sigma_1 > \sigma_2 > \sigma_3$ define three maximum shear stresses τ_1, τ_2, τ_3 in planes at 45° relative to subsequent sets of three principal stress directions. Can the hydrostatic stress be computed from these maximum shear stresses?

b. Show that in the geometrically linear theory, the deformation gradient F_{ij} can be expressed as

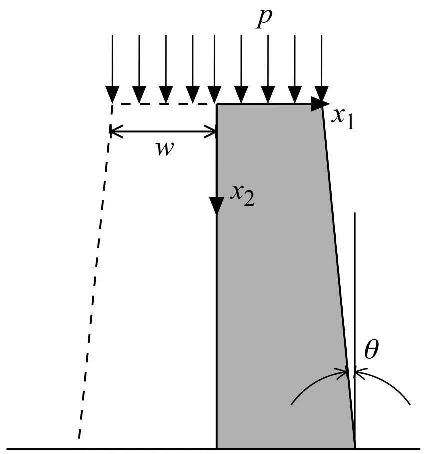
$$F_{ij} = \delta_{ij} + \varepsilon_{ij}$$

in case there are no rotations.

c. Consider the planar deformation field specified by the strain components

$$[\varepsilon_{\alpha\beta}] = \begin{bmatrix} \varepsilon_1 & 0 \\ 0 & \varepsilon_2 \end{bmatrix},$$

but first draw a unit circle, ${X_1}^2 + {X_2}^2 = 1$, onto the material. Taking into consideration that $dx_i = F_{ij}dX_j$ also applies to finite length elements when F_{ij} is uniform (as in this question), determine the shape of the square after the above strain has been applied.



Question 2 A long ridge is cut out of a single crystal by means of focused ion beam milling. Because of this process the sides of the ridge are not perfectly vertical but have a small taper angle θ. The properties of the crystal are tested by subjecting the top to a pressure, p, which is assumed to be uniformly distributed over the top width w. Since the ridge is long, we can also assume that it deforms in plane strain perpendicular to the x_1–x_2 plane shown. Because of symmetry only half of the cross-section (the gray area) needs to be analyzed.

a. Specify the boundary conditions for the left and the right-hand side of the gray region in terms of stress components.

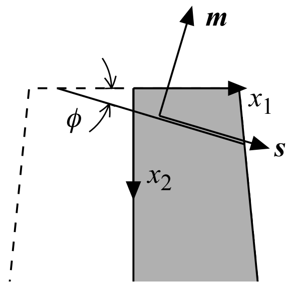
b. Assuming the vertical stress σ_{22} to be uniform in x_1, determine the value at arbitrary depth x_2.

c. Use equilibrium to find the differential equation for the shear stress σ_{12} in terms of σ_{22}. Solve this equation, taking into account the appropriate boundary conditions, and prove that the stress field satisfies

$$\frac{\sigma_{12}}{\sigma_{22}} = \frac{\xi_1}{1+\xi_2}, \quad \text{with } \xi_1 := (x_1/w)\tan\theta\,, \xi_2 := (x_2/w)\tan\theta$$

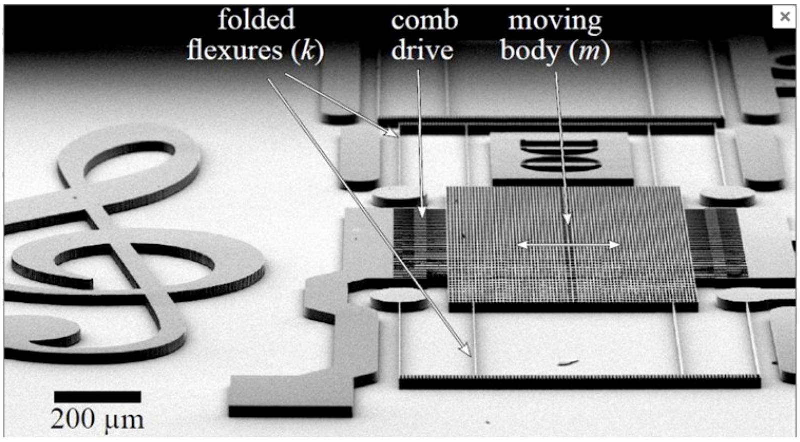
d. Similarly, find the differential equation for the shear stress σ_{11} in terms of σ_{12}. Solve this equation, taking into account the appropriate boundary conditions, and prove that σ_{11} varies quadratically with x_1.

e. Let the orientation of the slip plane normal $\boldsymbol{m}$ and the slip direction $\boldsymbol{s}$ of a slip system in the crystal be determined by the angle φ as indicated in this figure. At which location will slip initiate?

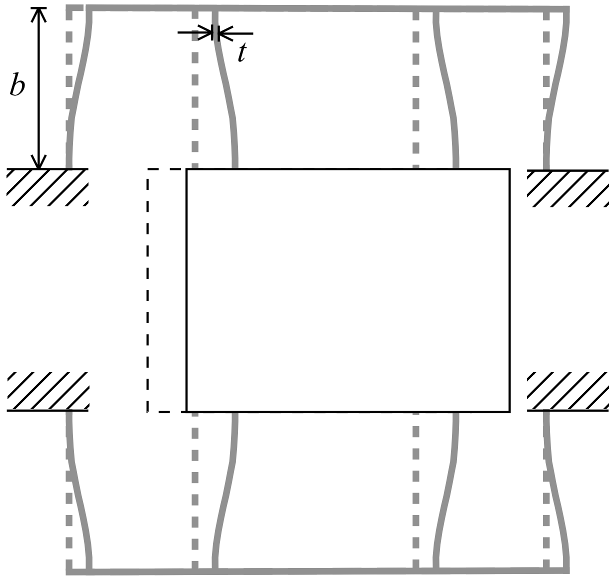


Question 3

A group of nanotechnology students from Twente University has recently presented a musical instrument called "micronium". It is special because it is able, by mechanical means, to produce tones with audible frequencies, despite the fact that the vibrating mass weighs only a few dozen micrograms. This is

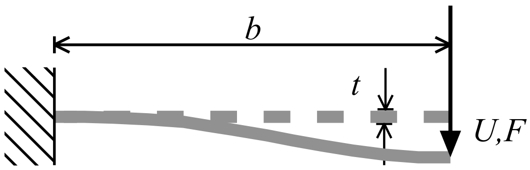


non-trivial in view of the well-known formula $f = \sqrt{k/m}/(2\pi)$ for the eigenfrequency of a mass (m)–spring system (stiffness k): at small scales, the mass m is necessarily small so that the stiffness needs to be small in order to keep f in the audible range ($\sim$ 20–20.000 Hz). The amplitude of the vibrations is only a few micrometers. The device has a separate unit for each tone; a scanning electron microscopy image of one unit is shown in the figure above. It consists of a thin rectangular plate that is mounted in between spring structures at the top and the bottom consisting of thin slender beams (the comb structures on the left and the right-hand side serve to actuate ('pluck') the mass-spring system and as a sensor to pick up of the frequency, which is then sent to an amplifier).



The figure on the right-hand side shows a mechanical model, comprising four interconnected leaf springs on top and bottom, having length b and width t (the thickness of the springs and the rectangular mass are the same). Both the equilibrium position (dashed lines) and a deflected configuration are shown (the outer leaf springs are mounted on posts fixed to the wafer).

a. First determine the stiffness F/U of a single leaf spring against a sideway displacement U, see figure below (this can be done either by solving the differential beam equation (3.49) plus appropriate boundary conditions or by combining the vergeet-me-nietjes of Fig. 3.6 in a clever way).



b. Denoting the stiffness determined above by k_1, compute the total stiffness k of the spring system in a micronium unit.

c. Using values of Young's modulus $E = 112\text{GPa}$ and mass density $\rho = 2.3\text{g}/\text{cm}^3$ for silicon, estimate the thickness t of the springs in order that the frequency of the tone produced by the micronium unit is 1000Hz. Note: from the micrograph at the start of this question, one could derive that $b = 200\ \mu\text{m}$ and that the moving mass has an area of $500 \times 600 = 30 \times 10^4\ \mu\text{m}^2$.

✍ ——— ✍

Question	# points
1	1.5
2	4.5
3	3.0

Grade = # points + 1

(1)

a) $\tau_1 = \frac{1}{2}(\sigma_1 - \sigma_2)$, ... etc.

the τ's are defined by differences of principal stresses → principal stresses σ_i's cannot be computed from τ_i's → <u>cannot</u> compute $\sigma_m = \frac{1}{3}(\sigma_1 + \sigma_2 + \sigma_3)$ from τ_i's

b) geometrically linear theory: $\varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i})$
no rotations: $\omega_{ij} = \frac{1}{2}(u_{i,j} - u_{j,i}) = 0$

$\Rightarrow \varepsilon_{ij} = u_{i,j}$

$\Rightarrow F_{ij} = \delta_{ij} + u_{i,j} = \delta_{ij} + \varepsilon_{ij}$

c) uniform F_{ij} means that all points (X_1, X_2) move to (x_1, x_2) given by $x_i = F_{ij} X_j$, that is

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1+F_{11} & 0 \\ 0 & 1+F_{22} \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = \begin{pmatrix} 1+\varepsilon_1 & 0 \\ 0 & 1+\varepsilon_2 \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \end{pmatrix}.$$

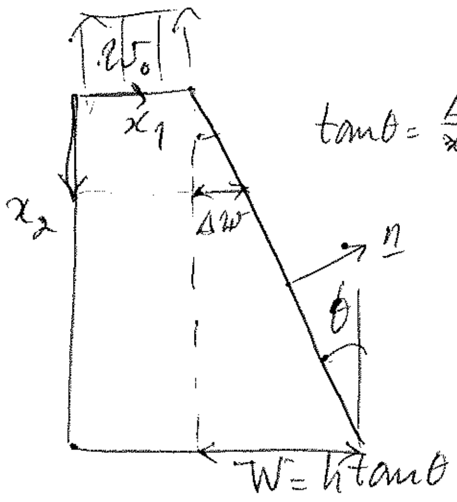
Points (X_1, X_2) on circle

$$X_1^2 + X_2^2 = \underbrace{\left(\frac{x_1}{1+\varepsilon_1}\right)^2 + \left(\frac{x_2}{1+\varepsilon_2}\right)^2 = 1}$$

defines <u>ellipse</u> with axes of length $1+\varepsilon_1$ and $1+\varepsilon_2$ in x_1 and x_2 direction, respectively.

(2)

$$\sigma^0 = F/w_0$$



$$\tan\theta = \frac{\Delta w}{x_2}$$

$$w(x_2) = w_0 + \Delta w = w_0 + x_2 \tan\theta$$

$$\sigma_{22}(x_2) = \frac{F}{w(x_2)} = \frac{F}{w_0(1 + x_2/w_0\, t)} = \frac{\sigma^0}{1 + \frac{x_2}{w_0} t}$$

$$\sigma_{22,2} = -\sigma^0\left(1 + \frac{x_2}{w_0} t\right)^{-2} t/w_0 = -\frac{t/w_0}{\left(1 + \frac{x_2}{w_0} t\right)}\,\sigma_{22}$$

$$\sigma_{12,1} + \sigma_{22,2} = 0 \Rightarrow \sigma_{12,1} = -\sigma_{22,2} = \sigma_{22}\,\frac{t/w_0}{1 + \frac{x_2}{w_0} t}$$

$$\Rightarrow \sigma_{12} = \sigma_{22}\left(\frac{\frac{x_1}{w_0} t}{1 + \frac{x_2}{w_0} t}\right) + C$$

$$n = \begin{pmatrix} c \\ -s \end{pmatrix} \Rightarrow \begin{pmatrix} \sigma_{11} c - \sigma_{12} s \\ \sigma_{12} c - \sigma_{22} s \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \text{ at } x_1 = w_0 + x_2 t = w_0\left(1 + \frac{x_2}{w_0} t\right),\ x_2$$

$$\cancel{\sigma_{12}}$$

$$\sigma_{12}\cancel{c} - \sigma_{22} t = \sigma_{22}\,\frac{\cancel{\left(1 + \frac{x_2}{w_0} t\right)}\, t}{\cancel{1 + \frac{x_2}{w_0} t}} + \cancel{C} - \sigma_{22} t = \underline{C = 0}.$$

$$\Rightarrow \sigma_{12}(x_1 = 0) = C = 0.$$

$$\frac{\sigma_{12}}{\sigma_0} = \frac{\left(\frac{x_1}{w}\right) t}{\left[1 + \left(\frac{x_2}{w}\right) t\right]^2}$$

$$\sigma_{11,1} + \sigma_{12,2} = 0$$

$$\sigma_{11,1} = -\sigma_{12,2} = -\left\{ \sigma_{22,2} \frac{\frac{x_1}{w} t}{1 + \frac{x_2}{w} t} - \sigma_{22} \frac{\frac{x_1}{w} t \ \frac{1}{w_0} t}{\left[1 + \frac{x_2}{w} t\right]^2} \right\}$$

$$= +\sigma_{22} \left\{ \frac{t/w_0 \ \frac{x_1}{w} t}{\left[1 + \frac{x_2}{w} t\right]^2} + \frac{\frac{x_1}{w} t \ \frac{1}{w} t}{[\qquad]^2} \right\}$$

$$= 2\, \sigma_{22} \frac{\frac{x_1}{w} t \ \frac{t}{w}}{\left[1 + \frac{x_2}{w} t\right]^2}$$

$$\sigma_{11} = \sigma_{22} \frac{\frac{x_1^2}{w^2} t^2}{\left[1 + \frac{x_2}{w} t\right]^2} + C_2$$

$$= \sigma_{12} \left(\frac{\frac{x_1}{w} t}{1 + \frac{x_2}{w} t} \right) + C_2$$

at $\left(x_1 = w_0\left(1 + \frac{x_2}{w_0} t\right), x_2\right)$: $\sigma_{11} = \sigma_{12} \tan\theta \Rightarrow$

$$\sigma_{12} \frac{\cancel{\left(1 + \frac{x_2}{w_0} t\right)} t}{\cancel{\left(1 + \frac{x_2}{w} t\right)}} = \left(\sigma_{12} \cancel{t} + C_2\right) t \Rightarrow C_2 = 0.$$

$$\frac{\sigma_{11}}{\sigma_0} = \frac{\left(\frac{x_1}{w}\right)^2 t^2}{\left[1 + \frac{x_2}{w} t\right]^3}$$

$$\tau = \sigma_{ij} P_{ij} = m_i \sigma_{ij} s_j$$

$$[m_i] = \begin{pmatrix} \sin\varphi \\ -\cos\varphi \end{pmatrix} \qquad [s_i] = \begin{pmatrix} \cos\varphi \\ +\sin\varphi \end{pmatrix}$$

$$\tau = \begin{pmatrix} s & -c \end{pmatrix} \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{12} & \sigma_{22} \end{pmatrix} \begin{pmatrix} c \\ s \end{pmatrix}$$

$$= \frac{1}{2}(\sigma_{11} - \sigma_{22}) \sin 2\varphi - \sigma_{12} \cos 2\varphi$$

All σ_{ij} largest at $\underline{x_2 = 0}$:

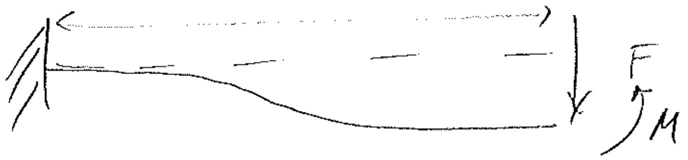
$$\frac{\sigma_{11}}{\sigma_0} = \left(\frac{x_1}{w} t\right)^2, \quad \frac{\sigma_{12}}{\sigma_0} = \overbrace{\left(\frac{x_1}{w} t\right)}^{\xi_1}, \quad \frac{\sigma_{22}}{\sigma_0} = 1$$

$$\Rightarrow \frac{\tau}{\sigma_0}(\xi_1) = \frac{1}{2}(\xi_1^2 - 1) \sin 2\varphi - \xi_1 \cos 2\varphi$$

$$0 = \frac{d\tau}{d\xi_1} = \xi_1 \sin 2\varphi - \cos 2\varphi \Rightarrow \underline{\xi_1 = \cot 2\varphi}$$

③

ⓐ

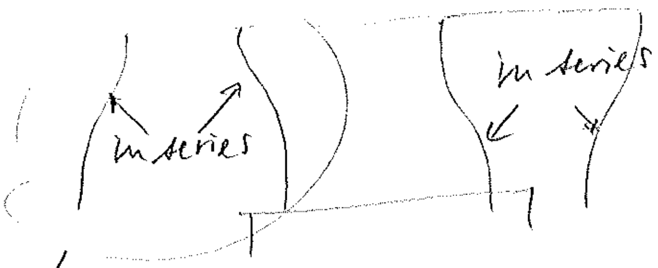


end rotation $= 0 = \dfrac{Fb^2}{2EI} - \dfrac{Mb}{EI} \Rightarrow \quad M = \frac{1}{2} F b$

end deflection $u = \dfrac{Fb^3}{3EI} - \dfrac{Mb^2}{2EI} = \left(\frac{1}{3} - \frac{1}{4}\right)\dfrac{Fb^3}{EI} = \dfrac{Fb^3}{12EI}$

$k_1 = F/u = \dfrac{12EI}{b^3}$, $I = \frac{1}{12} d t^3 \longrightarrow k_1 = Ed\left(\frac{t}{b}\right)^3$

ⓑ



$\dfrac{1}{k_2} = \dfrac{1}{k_1} + \dfrac{1}{k_1} = \dfrac{2}{k_1} \Rightarrow k_2 = \frac{1}{2} k_1$

top stiffness $= k_2 + k_2 = k_1$

btm " $\qquad k_1$

$k = 2k_1$

ⓒ $f = \dfrac{1}{2\pi}\sqrt{\dfrac{k}{m}} = \dfrac{1}{2\pi}\sqrt{\dfrac{24EI}{b^3}\dfrac{1}{\rho A d}}$

$= \dfrac{1}{2\pi}\sqrt{\dfrac{2E\cancel{d}t^3}{\rho A \cancel{d} b^3}} = \dfrac{1}{2\pi}\sqrt{\dfrac{2E}{\rho A}\left(\dfrac{t}{b}\right)^3}$

$\Rightarrow \dfrac{2E}{\rho A}\left(\dfrac{t}{b}\right)^3 = (2\pi f)^2 \Rightarrow \left(\dfrac{t}{b}\right)^3 = \dfrac{\rho A}{2E}(2\pi f)^2$

$= \dfrac{2.3\times 10^3 (kg/m^3)\ \cancel{25}\,30 \times 10^4 \times 10^{-12} (m^2)}{224 \times 10^9 (N/m^2)} \left(2\pi \cdot 10^3/s\right)^2$

$= \dfrac{2.3 \times \cancel{25}\,30}{224} (2\pi)^2 \times \dfrac{10^{3+4-12+6}}{10^9}$

$\Longleftarrow \quad = \cancel{10.73}\,12.16 \times 10^{-8} = 121.6 \times 10^{-9}$

$t \approx 4.95 \times 10^{-3} \cdot b$

$= 0.99\ \mu m$